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Weyl field strength symmetries for arbitrary helicity and gauge invariant Fierz–Pauli and Rarita–Schwinger wave equations

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Abstract. The algebra of the restricted Lorentz group and the Weyl formulation of massless Poincaré irreducible fields have been developed by primarily Lorentz-covariant Pauli matrix methods which facilitate the generation of both indexed Weyl spinor identities and Dirac identities. The results have been used to display a complete set of symmetries for the (anti)self-dual Weyl field strengths of arbitrary helicity $j(\geqslant 1)$. These symmetries and the requirement of Poincaré irreducibility have then been used to give a direct and uniform determination of the forms of the gauge invariant (Lagrangian) free field wave equations for the Fierz-Pauli and Rarita-Schwinger potentials of spin 1, $\frac{3}{2}$ and 2. The procedures set out indicate that it should be possible to establish the gauge invariant Lagrangian free field wave equations of arbitrary helicity in a uniform and direct manner from the corresponding much simpler equations governing the field strengths in unmixed spin representations.

1. Introduction

Fields which transform according to irreducible representations $D(j_1, j_2)$ of the restricted (inversion-free) Lorentz group $SO^+_{1,3}$ are customarily manipulated (van der Waerden 1929, Corson 1953, Penrose 1960, Misner *et al* 1973, Penrose and Rindler 1984) by expressing their components as explicitly indexed Weyl spinors $\psi_{A_1A_2\ldots A_{2j_1}\dot{U}_1\dot{U}_2\ldots\dot{U}_{2j_2}}$ $(A_1, A_2, \ldots, \dot{U}_1, \dot{U}_2, \ldots = 1, 2)$ which have complete symmetry on all undotted and complete symmetry on all dotted (complex conjugate) indices with indices raised and lowered using the antisymmetric bispinors $(\varepsilon^{AB}) = (\varepsilon_{\dot{U}\dot{V}}) = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ and their negative inverses $(\varepsilon_{AB}) = (\varepsilon^{\dot{U}\dot{V}}) = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. Many of the corresponding results for the rotation group, SO_3, and the full Lorentz group $O_{1,3}$, make use, on the contrary, of primarily matrix methods of the Pauli algebra on the one hand (Pauli 1927, Salingaros 1981, Salingaros and Ilamed 1984) and the Dirac algebra (Dirac 1928, Itzykson and Zuber 1980, van Nieuwenhuizen 1981) on the other. While, for many purposes, the indexed Weyl spinors and the Pauli-van der Waerden symbols $\sigma^{\mu A\dot{U}}$ (Corson 1953, Pirani 1965, Misner *et al* 1973, Penrose and Rindler 1984) are a powerful tool intimately linked to Lorentz irreducibility, in some instances matrix manipulations provide a complementary insight into the significance of the many identities involved. They can also appear less forbidding than the spinor-indexed counterparts.

We extend here the partly matrix treatments of Barut (1964), Lord (1976) and Wess and Bagger (1983) by supplying, in § 2.1 and the appendix, a more extensive list of

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Lorentz-covariant Pauli identities in matrix form. In §§ 2.2 and 2.3 we present Weyl's equations (Weyl 1928) in covariant Pauli matrix form and establish, *ab initio*, the properties of the two-component charge conjugation matrix in an arbitrary unitary representation of the Pauli algebra. We then apply some of these identities, in §§ 3, 4 and 5, to the Weyl fields of spin 1, $\frac{3}{2}$ and 2 to systematically and uniformly set out the relationships between the completely symmetric Weyl spinors and the corresponding gauge invariant tensor or tensor-spinor field strengths of the Maxwell, Rarita-Schwinger (gravitino) and Fierz-Pauli (graviton) fields. We then extend these spin 1, $\frac{3}{2}$ and 2 cases in § 6 to display the corresponding relations for the arbitrary spin Weyl fields and give a complete set of symmetries for the tensor and tensor-spinor field strengths confirming those given by Weinberg (1965) and Rodriguez and Lorente (1981, 1984). Finally, in § 7, we use the requirement of Poincaré irreducibility to provide a uniform direct determination and comparison of the gauge invariant Lagrangian field equations for the free field Fierz-Pauli and Rarita-Schwinger gauge potentials A_μ, ψ_μ and $h_{\mu\nu}$. A procedure for forming indexed Weyl spinor identities and for relating covariant Pauli and Dirac matrix identities is briefly outlined in the appendix.

2. Pauli identities and Weyl's equations

2.1. *Covariant Pauli algebra*

We let σ^k, $k = 1, 2, 3$, be the generating elements of any unitary matrix representation, $\sigma^{k+} = \sigma^k$, of the Pauli (Clifford) algebra with identity $\sigma^k\sigma^l = \mathrm{i}\varepsilon^{klm}\sigma^m + \delta^{kl}\mathbb{1}$ ($\sigma^k\sigma^l + \sigma^l\sigma^k = 2\delta^{kl}$) (Clifford 1878, Pauli 1927, Salingaros 1981, Salingaros and Ilamed 1984). With the Minkowski matrix $(\eta_{\mu\nu}) = \mathrm{diag}(+1, -1, -1, -1)$ we define invariant $\mathrm{SO}^{+}_{1,3}$ Pauli elements $\overset{+}{\sigma}{}^\mu = \{\mathbb{1}, \sigma^k\}$ and $\overset{+}{\sigma}_\mu = \eta_{\mu\nu}\overset{+}{\sigma}{}^\nu$ with their parity conjugates $\overset{-}{\sigma}{}^\mu = \{\mathbb{1}, -\sigma^k\}$ which, although they contain the same SO_3 Pauli elements σ^k as do $\overset{+}{\sigma}_\mu = \{\mathbb{1}, \sigma_k\} = \{\mathbb{1}, -\sigma^k\}$, require a separate definition due to the level of the index μ.

We define the invariant (anti)self-dual 'tensor' sets of Pauli elements $\overset{\pm}{\sigma}{}^{\mu\nu} = \frac{1}{2}\mathrm{i}(\overset{\mp}{\sigma}{}^\mu\overset{\pm}{\sigma}{}^\nu - \overset{\mp}{\sigma}{}^\nu\overset{\pm}{\sigma}{}^\mu) = \overset{\mp}{\sigma}{}^{\mu\nu+}$ which form direct and natural $\mathrm{SO}^{+}_{1,3}$ extensions of the SO_3 dual $\sigma^{kl} = \varepsilon^{klm}\sigma^m = \frac{1}{2}\mathrm{i}[\sigma^k(-\sigma^l) - \sigma^l(-\sigma^k)]$ of σ^m. Normalising the permutation symbol $\varepsilon^{\mu\nu\lambda\rho}$ and ε^{klm} to $\varepsilon^{0123} = +\varepsilon^{123} = +1 = -\varepsilon_{0123}$ we may now establish by straightforward algebraic manipulation the identities listed in the appendix. Equation (A13) in particular shows that $\frac{1}{2}\overset{\pm}{\sigma}{}^{\mu\nu}$ each satisfy the Lie algebra so(1, 3) of the restricted Lorentz group. An evaluation of the Casimir invariants $\boldsymbol{M}^2$ and $\boldsymbol{N}^2$ of the two su(2) subalgebra generators $\boldsymbol{M} = \frac{1}{2}(\boldsymbol{S} + \mathrm{i}\boldsymbol{K})$ and $\boldsymbol{N} = \frac{1}{2}(\boldsymbol{S} - \mathrm{i}\boldsymbol{K})$, where $S^k = \frac{1}{2}\varepsilon^{klm}S^{lm}$ and $K^m = S^{0m}$ generate the rotations and boosts, shows that $\frac{1}{2}\overset{-}{\sigma}{}^{\mu\nu}$ and $\frac{1}{2}\overset{+}{\sigma}{}^{\mu\nu}$ generate, respectively, the $D(\frac{1}{2}, 0)$ spin-$\frac{1}{2}$ irrep of $\mathrm{SO}^{+}_{1,3}$ and the conjugate $D(0, \frac{1}{2})$ irrep. The $\mathrm{SO}^{+}_{1,3}$ invariants $\overset{\pm}{\sigma}{}^\mu$ may, of course, be referred to as being Lorentz covariant parity conjugates as a result of, for example, the $\mathrm{O}_{1,3}$ transformation properties of $\psi^{+}_{\pm}\overset{\mp}{\sigma}{}^\mu\psi_\pm$ where ψ_- and ψ_+ are $D(\frac{1}{2}, 0)$ and $D(0, \frac{1}{2})$ fields.

2.2. *Weyl's equations of arbitrary spin*

Weyl's equations for negative and positive helicity $\lambda = \pm j$ ($j \geqslant \frac{1}{2}$) fields $\psi_- = (\chi^{A_1A_2\ldots A_{2j}})$ and $\psi_+ = (\bar{\phi}_{\dot{U}_1\dot{U}_2\ldots\dot{U}_{2j}})$ of the $D(j, 0)$ and $D(0, j)$ representations take the form $\not{\partial}_\pm\psi_\pm = 0$, where $\not{\partial}_\pm \equiv \overset{\pm}{\sigma}{}^\mu\partial_\mu$. These are in fact the only first-order, linear, restricted Poincaré covariant ($\mathrm{ISO}^{+}_{1,3}$) and parity-conjugate equations for $\psi_\pm$ separately. The Poincaré covariance follows as a consequence of the identities $\not{\partial}_\pm\not{\partial}_\mp \equiv \Box$, where $\Box = \partial^\mu\partial_\mu$, which give mass irreducibility (with mass zero) and the demonstration that $\psi_\pm$ are indeed spin-j fields. The latter is a consequence of their transformation properties as symmetrised products of $2j$ factors of $D(\frac{1}{2}, 0)$ or $D(0, \frac{1}{2})$ spinors. Indeed, substitution of $S^{\mu\nu} = \frac{1}{2}\overset{\pm}{\sigma}{}^{\mu\nu}$ into the Pauli–Lubański spin 4-vector $W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\lambda\rho} S_{\nu\lambda} P_\rho$ gives $W^\mu \psi_\pm = \pm\frac{1}{2}(\eta^{\mu\nu} - \overset{\mp}{\sigma}{}^\mu \overset{\pm}{\sigma}{}^\nu) P_\nu \psi_\pm$. Since the helicity operator is $W^0/|P^0| = \boldsymbol{S} \cdot \hat{\boldsymbol{P}}$ where $S^k = \frac{1}{2}\varepsilon^{klm} S^{lm}$, use of the field equations gives $\boldsymbol{S} \cdot \hat{\boldsymbol{P}} \psi_\pm = \pm\frac{1}{2}\varepsilon_P \psi_\pm$ for spin $\frac{1}{2}$ and $\boldsymbol{S} \cdot \hat{\boldsymbol{P}} \psi_\pm = \pm j \varepsilon_P \psi_\pm$ in general, where $\varepsilon_P = \pm 1$ applies to positive or negative energy contributions to $\psi_\pm$. The fields $\psi_\pm$ are therefore indeed massless helicity eigenstates and consequently Poincaré covariant. Using the Weyl representation of the chiral Dirac matrix $\gamma_5 = \left(\begin{smallmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{smallmatrix}\right)$, with $\psi = \left[\begin{smallmatrix} \psi_+ \\ \psi_- \end{smallmatrix}\right]$, the above relations become, in Dirac form, $\boldsymbol{S} \cdot \hat{\boldsymbol{P}} \psi = j \varepsilon_P \gamma_5 \psi$ showing that the helicity and chirality eigenstates correspond in the massless case.

The spin-$\frac{1}{2}$ fields are obtainable from the Hermitian Lagrangians $L_\pm = \frac{1}{2}\mathrm{i}\psi_\pm^\dagger \overleftrightarrow{\not{\partial}}_\pm \psi_\pm$ but for higher spin ($j \geqslant 1$) the Weyl fields cannot be obtained from Hermitian Lagrangians without the introduction of auxiliary fields (Guralnik and Kibble 1965, Larsen and Repko 1978). This may be readily seen by using explicitly indexed spinors. Consider, for example, the equation $\not{\partial}_- \psi_- = 0$, namely $\sigma^\mu_{A\dot{U}} \partial_\mu \chi^{AA_2 \ldots A_{2j}} = 0$. Since the equation is Poincaré covariant, first order and linear, a Lagrangian, if it exists, must be a real Poincaré invariant bilinear in $\partial_\mu \psi_-$ and the adjoint (or Hermitian conjugate) spinor $\psi_-^\dagger$ and/or in $\partial_\mu \psi_-^\dagger$ and ψ_-. It must therefore be of the form

$$\mathrm{i}\bar{\chi}^{\dot{U}_1 \dot{U}_2 \ldots \dot{U}_{2j}} \Sigma^\mu_{A_1 A_2 \ldots A_{2j} \dot{U}_1 \dot{U}_2 \ldots \dot{U}_{2j}} \overleftrightarrow{\partial}_\mu \chi^{A_1 A_2 \ldots A_{2j}}$$

where Σ^μ must be completely symmetric in all undotted and all dotted indices and must be Hermitian. Such a Lagrangian has Euler–Lagrange equations

$$\Sigma^\mu_{A_1 A_2 \ldots A_{2j} \dot{U}_1 \dot{U}_2 \ldots \dot{U}_{2j}} \partial_\mu \psi_-^{A_1 A_2 \ldots A_{2j}} = 0$$

which coincide with the original Weyl's equations if and only if $j = \frac{1}{2}$ and $\Sigma^\mu = \bar{\sigma}^\mu$. A similar analysis applies to the conjugate spinor ψ_+ and to the formalism of Bargmann and Wigner (1948) where an $\mathrm{IO}_{1,3}$ invariant Lagrangian must be sought and the mass of the field may also be non-zero.

2.3. *The Weyl charge conjugation matrix*

Following van Nieuwenhuizen (1981) for the Dirac case, we establish here the existence and properties of the two-component charge conjugation matrix ε without reference to any explicit representation of the Pauli algebra. The matrices σ_k^T or $-\sigma_k^T$ also satisfy the Pauli algebra and σ_k are an irreducible set of matrices. Schur's lemma (Wigner 1959, Butler 1981) implies, however, that there is only one inequivalent two-dimensional representation of the finite (Pauli) group generated by σ_k or $-\sigma_k^T$ and thus guarantees the existence of a matrix ε establishing their equivalence, which may be expressed in the form $\varepsilon^{-1}\sigma_k\varepsilon = -\sigma_k^T$ implying $\varepsilon^{-1}\overset{+}{\sigma}_\mu\varepsilon = \bar{\sigma}_\mu^T$. Moreover, since $\varepsilon^T\varepsilon^{-1}\sigma_k\varepsilon(\varepsilon^{-1})^T = \sigma_k$ follows by two applications of the equivalence relation, $\varepsilon^T\varepsilon^{-1}$ commutes with all σ_k and, again by Schur's lemma, is a multiple of the identity. Since $\varepsilon^T = k\varepsilon$ gives $k^2 = 1$ and $k = \pm 1$ we need only note that $(\sigma_k\varepsilon)^T = -k\sigma_k\varepsilon$ to conclude that $k = -1$ since otherwise three of the four linearly independent 2×2 matrices ε and $\sigma_k\varepsilon$ would be antisymmetric which is impossible. Consequently, ε is antisymmetric and $\sigma_k\varepsilon$ symmetric. It also follows directly that $\varepsilon^{-1}\overset{\pm}{\sigma}_{\mu\nu}\varepsilon = -\overset{\pm}{\sigma}{}_{\mu\nu}^T$. Application of Schur's lemma again shows that $\varepsilon^\dagger\varepsilon$ is a (positive) multiple of the identity allowing ε to be normalised to make it unitary. Finally, provided we choose representations in which each σ^k is either symmetric or antisymmetric, then a further application of Schur's lemma establishes that ε^2 can be chosen proportional to the identity, thus permitting the phase to be chosen so that ε is real and orthogonal. Analogously to the Dirac case, we see that, while ε is antisymmetric, $\overset{\pm}{\sigma}{}^k\varepsilon$ and $\overset{\pm}{\sigma}{}^{\mu\nu}\varepsilon$ are all symmetric. We therefore have the following representation-independent charge conjugation matrix properties:

$$\varepsilon^{-1}\overset{\pm}{\sigma}_\mu\varepsilon = \overset{\mp}{\sigma}{}^T_\mu \qquad \varepsilon^{-1}\overset{\pm}{\sigma}_{\mu\nu}\varepsilon = -\overset{\pm}{\sigma}{}^T_{\mu\nu} \qquad \varepsilon^T = -\varepsilon \qquad \varepsilon^\dagger\varepsilon = \mathbb{1} \tag{1}$$

whose solution is $\varepsilon = \mathrm{e}^{\mathrm{i}\theta}\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ with θ a real phase and the representation-dependent results

$$\varepsilon = \varepsilon^* \qquad (\Rightarrow \varepsilon = -\varepsilon^{-1},\ \varepsilon^T\varepsilon = \mathbb{1}) \tag{2}$$

whose only solutions for ε require $\theta = 0, \pi$. Apart from always assuming a real charge conjugation matrix we shall never require an explicit representation of the elements σ^k of the Pauli algebra. However, if we choose the standard Pauli matrices

$$\sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma^2 = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix} \qquad \sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{3}$$

to represent σ^k, then since they are each either symmetric or antisymmetric we are guaranteed the existence of a real charge conjugation matrix which is conventionally taken to be $\varepsilon = \mathrm{i}\sigma^2 = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ (the only other real orthogonal antisymmetric choice being $-\mathrm{i}\sigma^2$).

Finally, to justify the name 'charge conjugation matrix', we note that if ψ_+ were a right-handed $D(0, \frac{1}{2})$ spinor of charge q coupled minimally to some 4-vector or pseudo-4-vector field A^μ according to $(\mathrm{i}\partial\!\!\!/_+ + qA\!\!\!/)\psi_+ = 0$, then it follows that $\psi_- = (\psi_+)_{\mathrm{conj}} = \varepsilon\psi_+^*$ will satisfy $(\mathrm{i}\partial\!\!\!/_- + (-q)A\!\!\!/)\psi_- = 0$ with the sign change q to $-q$ indicating that ψ_- will be oppositely charged. Similarly, the identity $\varepsilon^{-1}\overset{\pm}{\sigma}_{\mu\nu}\varepsilon = -\overset{\pm}{\sigma}{}^T_{\mu\nu}$ will ensure that ψ_- transforms by $D(\frac{1}{2}, 0)$ if ψ_+ transforms by $D(0, \frac{1}{2})$. A Weyl particle cannot be electrically charged but a similar argument will apply to any other charge such as lepton number for neutrinos for example.

We next apply the covariant Pauli matrix identities successively to the spin-1, $\frac{3}{2}$ and 2 Weyl spinors to determine the properties of the corresponding tensor and tensor–spinor field strengths.

3. Spin 1: Maxwell field

Let $\psi_- = (\chi^{AB})$ and $\psi_+ = (\bar{\phi}_{\dot{U}\dot{V}})$ be symmetric matrices representing $D(1, 0)$ and $D(0, 1)$ spinors each with three independent complex components. We form the complex tensors

$$F^\pm_{\mu\nu} = \tfrac{1}{4}\,\mathrm{Tr}(\psi_\pm\varepsilon^{\mp 1}\overset{\pm}{\sigma}_{\mu\nu}) \tag{4}$$

(one of which would be $F^+_{\mu\nu} = -\frac{1}{4}\bar{\phi}_{\dot{U}\dot{V}}\varepsilon^{\dot{V}\dot{W}}\overset{+}{\sigma}_{\mu\nu\dot{W}}{}^{\dot{U}} = -\frac{1}{4}\bar{\phi}_{\dot{U}\dot{V}}\overset{+}{\sigma}_{\mu\nu}{}^{\dot{V}\dot{U}}$ in spinor-indexed notation). Equations (A12) and (A15) ensure the antisymmetry and (anti)self-duality of $F^\pm_{\mu\nu}$, namely $F^\pm_{[\mu\nu]} = F^\pm_{\mu\nu}$ and $\mathrm{i}\tilde{F}^\pm_{\mu\nu} = \pm F^\pm_{\mu\nu}$ where the dual is $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$ leaving each with three complex components. These symmetries and identities (A23) establish the inverse relations

$$\psi_\pm = \tfrac{1}{2}F^\pm_{\mu\nu}\overset{\pm}{\sigma}{}^{\mu\nu}\varepsilon^{\pm 1} \tag{5}$$

demonstrating that $\psi_\pm$ and $F^\pm_{\mu\nu}$ contain the same information and thus showing that we have all the symmetries of $F^\pm_{\mu\nu}$. Equation (A19) provides the dependent symmetries $\overset{\pm}{\sigma}{}^{\mu\nu}F^\mp_{\mu\nu} = 0$.

From $\overset{\pm}{\sigma}{}^{+}_{\mu\nu}=\overset{\mp}{\sigma}_{\mu\nu}$ and $(\overset{\pm}{\sigma}{}^{\mu\nu}\varepsilon)^T=\overset{\pm}{\sigma}{}^{\mu\nu}\varepsilon$ we find that the conjugate $\varepsilon\psi_+^*\varepsilon^T$ of the spin-1 spinor ψ_+ is related to the complex conjugate of $F^+_{\mu\nu}$ according to $\varepsilon\psi_+^*\varepsilon^T=\frac{1}{2}F^{+*}_{\mu\nu}\bar{\sigma}^{\mu\nu}\varepsilon^{-1}$, in the same way that ψ_- is related to $F^-_{\mu\nu}$. Consequently, if $\psi_\pm$ are mutually conjugate, $\psi_-=\varepsilon\psi_+^*\varepsilon^T$, then $F^\pm_{\mu\nu}$ are complex conjugates and $F_{\mu\nu}=F^-_{\mu\nu}+F^+_{\mu\nu}$ is a real tensor from which $\psi_\pm$ may be retrieved, using $\overset{\pm}{\sigma}{}^{\mu\nu}F^\mp_{\mu\nu}=0$, in the form

$$\psi_\pm=\tfrac{1}{2}F_{\mu\nu}\overset{\pm}{\sigma}{}^{\mu\nu}\varepsilon^{\pm1}. \tag{6}$$

$F_{\mu\nu}$ therefore transforms $\mathrm{O}_{1,3}$ irreducibly according to the real $\mathrm{SO}^+_{1,3}$ reducible representation $D(1,0)\oplus D(0,1)$. The field equations for $F^\pm_{\mu\nu}$ are determined by noting that equation (A14) gives $\partial^\mu F^\pm_{\mu\nu}=\frac{1}{4}\mathrm{i}\,\mathrm{Tr}[\varepsilon^{\mp1}(\partial_\nu\psi_\pm-\overset{\mp}{\sigma}_\nu\overset{\pm}{\partial}\psi_\pm)]$ which vanishes by the antisymmetry of $\varepsilon^{\mp1}$, the symmetry of $\psi_\pm$ and the Weyl equations, giving $\partial^\mu F^\pm_{\mu\nu}=0$, $\partial^\mu F_{\mu\nu}=0=\partial^\mu\tilde{F}_{\mu\nu}$ and $\Box F_{\mu\nu}=0$. Thus $F_{\mu\nu}$ clearly has all the properties of the free Maxwell field strength, namely those of the unique real non-Lagrangian irreducible Poincaré representation of helicity 1.

4. Spin $\frac{3}{2}$: massless Rarita–Schwinger field (gravitino)

$\psi_-=(\chi^{ABC})$ and $\psi_+=(\bar{\phi}_{\dot U\dot V\dot W})$ are now taken to be completely symmetric $D(\frac{3}{2},0)$ and $D(0,\frac{3}{2})$ Weyl spinors, each with four independent complex components. We form the antisymmetric (anti)self-dual tensor-spinor field strengths:

$$f^\pm_{\mu\nu}=\tfrac{1}{4}\mathrm{Tr}(\psi_\pm\varepsilon^{\mp1}\overset{\mp}{\sigma}_{\mu\nu}) \tag{7}$$

with inverses

$$\psi_\pm=\tfrac{1}{2}f^\pm_{\mu\nu}\otimes\overset{\pm}{\sigma}{}^{\mu\nu}\varepsilon^{\pm1}. \tag{8}$$

Now $\overset{\pm}{\sigma}{}^{\mu\nu}f^\pm_{\mu\nu}=\mathrm{Tr}(\overset{\pm}{\sigma}{}^{\mu\nu}\otimes f^\pm_{\mu\nu})=2\,\mathrm{Tr}(\psi_\pm\varepsilon^{\mp1})=0$ where we have used the complete symmetry of $\psi_\pm$ and identity (A25). This leaves four independent components in each $f^\pm_{\mu\nu}$ and shows that we have found a complete set of symmetries. An important dependent symmetry may also be established:

$$\begin{aligned}\overset{\pm}{\sigma}_\mu f^{\pm\mu\nu}&=\pm\tfrac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\lambda\rho}\overset{\pm}{\sigma}_\mu f^\pm_{\lambda\rho} &&\text{by (anti)self-duality}\\ &=\tfrac{1}{2}(\eta^{\nu\lambda}\overset{\pm}{\sigma}{}^\rho-\eta^{\nu\rho}\overset{\pm}{\sigma}{}^\lambda-\mathrm{i}\overset{\pm}{\sigma}{}^{\nu}\overset{\pm}{\sigma}{}^{\lambda\rho})f^\pm_{\lambda\rho} &&\text{by (A16)}\\ &=-\overset{\pm}{\sigma}_\lambda f^{\pm\lambda\nu}\end{aligned}$$

which must therefore vanish.

One may now establish that

$$f^\pm_{[\mu\nu]}=f^\pm_{\mu\nu}\qquad \mathrm{i}\tilde{f}^\pm_{\mu\nu}=\pm f^\pm_{\mu\nu}\qquad \overset{\pm}{\sigma}{}^{\mu\nu}f^\pm_{\mu\nu}=0 \tag{9}$$

and

$$f^\pm_{[\mu\nu]}=f^\pm_{\mu\nu}\qquad \overset{\pm}{\sigma}{}^\mu f^\pm_{\mu\nu}=0 \tag{10}$$

are alternate complete sets of independent symmetries for $f^\pm_{\mu\nu}$. The Weyl equations now show that $\partial^\mu f^\pm_{\mu\nu}=0=\partial^\mu\tilde{f}^\pm_{\mu\nu}$ analogously to Maxwell's equations. A direct sum can also be used to construct the Dirac tensor-spinor field strength

$$f_{\mu\nu}=\begin{bmatrix}f^+_{\mu\nu}\\ f^-_{\mu\nu}\end{bmatrix}$$

of the spin-$\frac{3}{2}$ field of Rarita and Schwinger (1941). With a self-conjugacy condition, $f^-_{\mu\nu}=\varepsilon f^{+*}_{\mu\nu}$ or $f_{\mu\nu}=C\bar{f}^T_{\mu\nu}$, this becomes a Majorana tensor-spinor appropriate to a gravitino field strength (Majorana 1937, van Nieuwenhuizen 1981).

5. Spin 2: massless Fierz–Pauli field (linearised gravity)

One new feature occurs at spin 2, after which it will be possible to formulate the arbitrary spin case. Let $\psi_- = (\chi^{ABCD})$ and $\psi_+ = (\bar{\phi}_{\dot{U}\dot{V}\dot{W}\dot{X}})$ be completely symmetric Weyl spinors transforming as $D(2,0)$ and $D(0,2)$ irreps of $SO^+_{1,3}$. We form the complex tensors

$$C^{\pm}_{\mu\nu\lambda\rho} = \tfrac{1}{16}\,\mathrm{Tr}(\mathrm{Tr}(\psi_\pm \varepsilon^{\mp 1} \overset{\pm}{\sigma}_{\mu\nu})\varepsilon^{\mp 1}\overset{\pm}{\sigma}_{\lambda\rho}) \tag{11}$$

with inverses

$$\psi_\pm = \tfrac{1}{4} C^{\pm}_{\mu\nu\lambda\rho}\overset{\pm}{\sigma}{}^{\mu\nu}\varepsilon^{\pm 1} \otimes \overset{\pm}{\sigma}{}^{\lambda\rho}\varepsilon^{\pm 1}. \tag{12}$$

The symmetries and duality properties

$$C^{\pm}_{\mu\nu\lambda\rho} = C^{\pm}_{[\mu\nu][\lambda\rho]} = C^{\pm}_{\lambda\rho\mu\nu} \qquad \mathrm{i}C^{\pm}_{\widetilde{\mu\nu}\lambda\rho} = \mathrm{i}C^{\pm}_{\mu\nu\widetilde{\lambda\rho}} = \pm C^{\pm}_{\mu\nu\lambda\rho} \tag{13}$$

leave each $C^{\pm}_{\mu\nu\lambda\rho}$ with six complex components, compared with five for each $\psi_\pm$, showing that one complex constraint remains to be found. Using (A14) and the complete symmetry of $\psi_\pm$ we obtain

$$C^{\pm\lambda}_{\mu\lambda\nu} = \tfrac{1}{16}\,\mathrm{Tr}(\mathrm{Tr}(\psi_\pm \varepsilon^{\mp 1}\overset{\mp}{\sigma}_\mu \overset{\pm}{\sigma}{}^\lambda)\varepsilon^{\mp 1}\overset{\mp}{\sigma}_\lambda \overset{\pm}{\sigma}_\nu).$$

Careful use of (A11) now shows that $C^{\pm\lambda}_{\mu\lambda\nu} = 0$, which, on using the symmetries and duality properties, can be reduced to the single independent symmetry $C^{\pm}_{[\mu\nu\lambda\rho]} = 0$, which is the one remaining constraint required.

With $\psi_- = \varepsilon(\varepsilon\psi_+^* \varepsilon^T)\varepsilon^T$ conjugate to ψ_+ one may obtain a self-conjugate reducible representation $D(2,0) \oplus D(0,2)$ of $SO^+_{1,3}$ in the form of the real $O_{1,3}$ irrep $C_{\mu\nu\lambda\rho} = C^-_{\mu\nu\lambda\rho} + C^+_{\mu\nu\lambda\rho}$, having as a complete set of symmetries

$$C_{\mu\nu\lambda\rho} = C_{[\mu\nu][\lambda\rho]} = C_{\lambda\rho\mu\nu} \qquad C^{\lambda}{}_{\mu\lambda\nu} = 0 \qquad C_{[\mu\lambda\nu\rho]} = 0 \tag{14}$$

which are, of course, precisely those of a Weyl (or vacuum Riemann) tensor. $C_{\mu\nu\lambda\rho}$ is the gauge invariant field strength of the spin-2 massless field of Fierz and Pauli (1939) applicable to a linearised description of gravity (Weinberg 1972, Misner *et al* 1973). The symmetries $\overset{\pm}{\sigma}{}^{\mu\nu} C^{\mp}_{\mu\nu\lambda\rho} = 0$ show that each $\psi_\pm$ can then be obtained from $C_{\mu\nu\lambda\rho}$ according to

$$\psi_\pm = \tfrac{1}{4} C_{\mu\nu\lambda\rho}\overset{\pm}{\sigma}{}^{\mu\nu}\varepsilon^{\pm 1} \otimes \overset{\pm}{\sigma}{}^{\lambda\rho}\varepsilon^{\pm 1}. \tag{15}$$

From $\partial\!\!\!/_\pm \psi_\pm = 0$ we deduce $\partial^\mu C^{\pm}_{\mu\nu\lambda\rho} = 0$ and hence $\partial^\mu C_{\mu\nu\lambda\rho} = 0$, $\partial^\mu C_{\widetilde{\mu\nu}\lambda\rho} = \partial^\mu C_{\mu\nu\widetilde{\lambda\rho}} = 0$ and finally $\Box C_{\mu\nu\lambda\rho} = 0$ confirming the Poincaré covariance of $C_{\mu\nu\lambda\rho}$.

6. Spin *j*

Making use of the previous lower spin results it will now be possible to simply write down the corresponding relations for arbitrary spin.

6.1. *Integer spin*

$\psi_\pm = (\chi^{A_1 \ldots A_{2j}})$ and $\psi_+ = (\bar{\phi}_{\dot{U}_1 \ldots \dot{U}_{2j}})$ each have $(2j+1)$ independent complex components. The tensor field strengths

$$F^{\pm}_{\mu_1 \ldots \mu_{2j}} = (\tfrac{1}{4})^j\,\mathrm{Tr}(\ldots \mathrm{Tr}(\mathrm{Tr}(\psi_\pm \varepsilon^{\mp 1}\overset{\pm}{\sigma}_{\mu_1\mu_2})\varepsilon^{\mp 1}\overset{\pm}{\sigma}_{\mu_3\mu_4}) \ldots \varepsilon^{\mp 1}\overset{\pm}{\sigma}_{\mu_{2j-1}\mu_{2j}}) \tag{16}$$

have inverses

$$\psi_\pm = (\tfrac{1}{2})^j F^\pm_{\mu_1\ldots\mu_{2j}} \overset{\pm}{\sigma}{}^{\mu_1\mu_2} \varepsilon^{\pm 1} \otimes \ldots \otimes \overset{\pm}{\sigma}{}^{\mu_{2j-1}\mu_{2j}} \varepsilon^{\pm 1} \tag{17}$$

showing that each contains the same information. $F^\pm_{\mu_1\ldots\mu_{2j}}$ are clearly antisymmetric and (anti)self-dual on each successive pair of indices and symmetric on permutations of the pairs, leaving each with $(j+1)(j+2)/2$ independent complex components, which shows that a further $j(j-1)/2$ independent constraints exist for each. Provided $j \geqslant 2$, the trace on any pair of indices, each from a different antisymmetric pair, vanishes:

$$F^{\pm\lambda}{}_{\mu_2\ldots\lambda\ldots\mu_{2j}} = 0.$$

This reduces to $F^\pm_{\mu[\nu\lambda\rho]\ldots} = 0$ and hence to $F^\pm_{[\mu\nu\lambda\rho]\ldots} = 0$ using the earlier symmetries and thus we have the remaining $j(j-1)/2$ constraints and a complete independent set.

As in the spin-1 and 2 cases, $\overset{\pm}{\sigma}{}^{\mu\nu} F^\mp_{\mu\nu\ldots} = 0$. Hence a real $D(j,0) \oplus D(0,j)$ tensor $F_{\mu_1\ldots\mu_{2j}} = F^-_{\mu_1\ldots\mu_{2j}} + F^+_{\mu_1\ldots\mu_{2j}}$ can be formed from the mutually conjugate $\psi_\pm$ which can be retrieved by

$$\psi_\pm = (\tfrac{1}{2})^j F_{\mu_1\ldots\mu_{2j}} \overset{\pm}{\sigma}{}^{\mu_1\mu_2} \varepsilon^{\pm 1} \otimes \ldots \otimes \overset{\pm}{\sigma}{}^{\mu_{2j-1}\mu_{2j}} \varepsilon^{\pm 1}. \tag{18}$$

A complete independent set of symmetries for $F_{\mu_1\ldots\mu_{2j}}$ is antisymmetry in each successive pair, symmetry under permutation of the pairs, tracelessness across pairs and a vanishing completely antisymmetric part on, for example, the first four indices, $F_{[\mu\nu\lambda\rho]\ldots} = 0$, with $F_{[\mu\nu\lambda]\rho\ldots} = 0$ a useful dependent symmetry. The Weyl equations establishing $\psi_\pm$ as Poincaré irreps similarly imply that

$$\partial^\mu F^\pm_{\mu\nu\lambda\rho\ldots} = 0 \qquad \partial^\mu F^\pm_{\widetilde{\mu\nu}\lambda\rho\ldots} = \partial^\mu F^\pm_{\mu\nu\widetilde{\lambda\rho}\ldots} = \ldots = 0 \tag{19}$$

and consequently $\partial_{[\mu} F^\pm_{\nu\lambda]\ldots} = 0$ and $\Box F^\pm_{\mu_1\ldots\mu_{2j}} = 0$ with similar results for the real field $F_{\mu_1\ldots\mu_{2j}}$.

6.2. *Half odd integer spin*

We write $j = n + \frac{1}{2}$ with n integral ($\geqslant 0$) and define tensor-spinor field strengths by

$$f^\pm_{\mu_1\ldots\mu_{2n}} = (\tfrac{1}{4})^n \operatorname{Tr}(\ldots \operatorname{Tr}(\operatorname{Tr}(\psi_\pm \varepsilon^{\mp 1} \overset{\pm}{\sigma}_{\mu_1\mu_2}) \varepsilon^{\mp 1} \overset{\pm}{\sigma}_{\mu_3\mu_4}) \ldots \varepsilon^{\mp 1} \overset{\pm}{\sigma}_{\mu_{2n-1}\mu_{2n}} \tag{20}$$

with inverses

$$\psi_\pm = (\tfrac{1}{2})^n f^\pm_{\mu_1\ldots\mu_{2n}} \otimes \overset{\pm}{\sigma}{}^{\mu_1\mu_2} \varepsilon^{\pm 1} \otimes \ldots \otimes \overset{\pm}{\sigma}{}^{\mu_{2n-1}\mu_{2n}} \varepsilon^{\pm 1}. \tag{21}$$

The tensor-spinors $f^\pm_{\mu_1\ldots\mu_{2n}}$ satisfy the same tensor index symmetries as those in the integer spin case, which reduces the number of independent complex components for each value of the spinor index to $2n+1$ for each value of the spinor index or $2(2n+1)$ in all, indicating that $f^\pm_{\mu\nu}$ satisfy a further $2n$ complex conditions each. As in the spin-$\frac{3}{2}$ case these final independent spinor trace symmetries are $\overset{\pm}{\sigma}{}^{\mu\nu} f^\pm_{\mu\nu\ldots} = 0$ which, with the tensor symmetries, give a complete set. These symmetries and the duality properties may be replaced by the spinor trace conditions $\overset{\pm}{\sigma}{}^\mu f^\pm_{\mu\nu\ldots} = 0$ to give an alternate complete set of independent symmetries. The Weyl equations imply field equations similar to those of the integer spin case.

The symmetries displayed here for the Weyl field strengths of arbitrary spin correspond to those given by Rodriguez and Lorente (1981, 1984) using the Dirac formalism, for chiral and (anti)self-dual Bargmann–Wigner fields.

7. The gauge invariant Lagrangian wave equations of spin 1, $\frac{3}{2}$ and 2

A further advantage of the procedures we have developed in the preceding sections is that they permit one to determine, directly and uniformly, from the Poincaré fields based on the unmixed Lorentz representations, the form that must be taken by the free field Lagrangian wave equations for the mixed Lorentz representation spin-1, $\frac{3}{2}$ and 2 gauge potentials A_μ, ψ_μ and $h_{\mu\nu}$ in order for them to irreducibly represent the Poincaré group. The Maxwell, Rarita-Schwinger and Fierz-Pauli field strengths $F_{\mu\nu}$,

$$f_{\mu\nu} = \begin{bmatrix} f^{+}_{\mu\nu} \\ f^{-}_{\mu\nu} \end{bmatrix}$$

and $C_{\mu\nu\lambda\rho}$ have full ($IO_{1,3}$) Poincaré covariance (including inversions). Since they are equivalent to the potentials, modulo gauge freedom, the wave equations for the latter, obtained by integration using the Poincaré lemma (Spivak 1965, Brittin *et al* 1982), will also be Poincaré covariant.

One application of the lemma to $F_{\mu\nu}$ establishes the existence of the potential A_μ satisfying $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the wave equation $\Box A^\mu - \partial^\mu \partial \cdot A = 0$. For comparison with $j = \frac{3}{2}$ and 2, we recall that this equation is invariant under the gauge transformation $\delta A_\mu = \partial_\mu \xi$ (ξ a scalar field), has an identically divergence-free left-hand side (which on the field strength takes the form of the source constraint $\partial_\mu \partial_\nu F^{\mu\nu} \equiv 0$) and is derivable from a Hermitian Lagrangian, facilitating coupling to other fields with conserved currents.

We now note that, by using Dirac matrices in the chiral representation,

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \overset{+}{\sigma}{}^\mu & 0 \end{pmatrix}$$

the spin-$\frac{3}{2}$ Weyl symmetries $\overset{\pm}{\sigma}{}^\mu f^{\pm}_{\mu\nu} = 0$ become the gamma trace symmetry $\gamma^\mu f_{\mu\nu} = 0$. Application of the lemma to the spin-$\frac{3}{2}$ field strength $f_{\mu\nu}$ establishes the existence of a vector-spinor potential ψ_μ which satisfies $f_{\mu\nu} = \partial_\mu \psi_\nu - \partial_\nu \psi_\mu$ and, from $\partial_\mu f^{\mu\nu} = 0$, the second-order wave equation $\Box \psi_\mu - \partial_\mu \partial \cdot \psi = 0$. However, the trace condition $\gamma^\mu f_{\mu\nu} = 0$ supplies the basic first-order equation on ψ_μ (from which the second-order equation follows but not vice versa), namely $\not\partial \psi_\mu - \partial_\mu \gamma \cdot \psi = 0$, as required for a positive definite probability density and consistent second quantisation of a fermionic field. The second-order equation and its γ contraction $\Box \gamma \cdot \psi - \not\partial \partial \cdot \psi = 0$ are differential constraints on ψ_μ that only apply on-shell. In the original discussion of the spin-$\frac{3}{2}$ field (Rarita and Schwinger 1941), invariance under the fermionic gauge transformation $\delta\psi_\mu = \partial_\mu \varepsilon$ with ε a spin-$\frac{1}{2}$ spinor field was noted as a property of the wave equation. This is also true for the above equation, surely the simplest of the many free field forms of the Rarita-Schwinger equation for the gravitino or supersymmetric partner of the graviton in supergravity theory (Freedman and van Nieuwenhuizen 1976, van Nieuwenhuizen 1981).

However, in constrast to the spin-1 case, the above equation, $\not\partial \psi_\mu - \partial_\mu \gamma \cdot \psi = 0$, does not have an off-shell divergence-free left-hand side and consequently cannot be derived from a gauge invariant action with Hermitian Lagrangian. This off-shell requirement on the variational derivative $\delta L/\delta\psi_\mu \equiv \partial L/\partial\psi_\mu - \partial_\lambda\, \partial L/\partial\partial_\lambda\psi_\mu$ (whose vanishing gives the Euler-Lagrange field equations) is established by considering a variation of the action, $\delta S = \int d^4x (\delta L/\delta\psi_\mu)\delta\psi_\mu$. Use of an arbitrary variation would now establish the field equations. Instead, one may use the presumed gauge invariance of the action ($\delta S = 0$ under $\delta\psi_\mu = \partial_\mu \varepsilon$) and integration by parts, discarding a surface term, to obtain $\int d^4x(\partial_\mu(\delta L/\delta\psi_\mu))\varepsilon=0$, and consequently the off-shell necessity of $\partial_\mu(\delta L/\delta\psi_\mu)\equiv 0$ for the field equations $\delta L/\delta\psi_\mu=0$ to be derivable from a gauge invariant action. Clearly the same argument applies to the spin-1 potential A_μ and to certain other gauge potentials with additional (internal or external) indices such as the Fierz–Pauli $h_{\mu\nu}$, general metric tensor $g_{\mu\nu}$ or Yang–Mills fields $B_\mu=(B^a_\mu)$.

Finally, two applications of the Poincaré lemma (Pirani 1965, p 279) to $C_{\mu\nu\lambda\rho}$ establish the existence of the spin-2 Fierz–Pauli potential $h_{\mu\nu}$, in terms of which the field strength is a double covariant curl $C_{\mu\nu\lambda\rho}=2\partial_{[\nu}h_{\mu][\lambda,\rho]}$. The Fierz–Pauli equation of second order for $h_{\mu\nu}$ arises from the trace condition on the field strength $C_{\mu\nu\lambda\rho}$:

$$2C^\lambda{}_{\mu\lambda\nu}\equiv 2R^L_{\mu\nu}\equiv -\Box h_{\mu\nu}+2\partial^\lambda\partial_{(\mu}h_{\nu)\lambda}-\partial_\mu\partial_\nu h=0 \tag{22}$$

where $h=h^\mu{}_\mu$. The first-order equation on the field strength $\partial^\mu C_{\mu\nu\lambda\rho}=0$ and its contraction, $\partial^\mu C^\lambda{}_{\mu\lambda\nu}=0$ or $\partial_\nu(\partial^\lambda\partial^\rho h_{\lambda\rho}-\Box h)=0$, become on-shell differential constraints on $h_{\mu\nu}$. As in the spin-$\frac{3}{2}$ case, this linearised Ricci form of the spin-2 equation (Fierz and Pauli 1939, Misner *et al* 1973) is invariant under the gauge transformation $\delta h_{\mu\nu}=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu$ but, as we have seen, does not have an off-shell divergence-free left-hand side and hence cannot be derived from a gauge invariant action nor coupled to a conserved source such as the matter or total energy-momentum tensor.

In order to see the need to self-couple in both the spin-$\frac{3}{2}$ and 2 cases and, in the case of spin $\frac{3}{2}$, to also couple to the spin-2 field supersymmetrically in order to achieve full consistency (Deser 1970, 1980, Boulware and Deser 1979) it is clearly necessary to achieve linear off-shell divergence freedom first. This may be achieved without loss of either gauge invariance or Poincaré covariance by subtracting from each equation a multiple of their own (on-shell vanishing) trace, namely $\frac{1}{2}\gamma_\mu$ times the spinorial trace, $2(\partial\cdot\psi-\not\partial\gamma\cdot\psi)$, and $\frac{1}{2}\eta_{\mu\nu}$ times the tensorial trace, $2(\partial^\lambda\partial^\rho h_{\lambda\rho}-\Box h)$, respectively. The results are the standard Lagrangian forms of, firstly, the massless Rarita–Schwinger equation,

$$\not\partial\psi_\mu-\partial_\mu\gamma\cdot\psi-\gamma_\mu\partial\cdot\psi+\gamma_\mu\not\partial\gamma\cdot\psi=0 \tag{23}$$

which has a great many other alternative forms (Freedman and van Nieuwenhuizen 1976, Wiltshire 1983), and the Einstein form of the Fierz–Pauli equation of spin 2:

$$2G^L_{\mu\nu}\equiv -\Box h_{\mu\nu}+2\partial^\lambda\partial_{(\mu}h_{\nu)\lambda}-\partial_\mu\partial_\nu h+\eta_{\mu\nu}\Box h-\eta_{\mu\nu}\partial^\lambda\partial^\rho h_{\lambda\rho}=0 \tag{24}$$

where $G^L_{\mu\nu}$ is the linearised Einstein tensor. Each of these free field equations is equivalent on-shell to the slightly simpler equations from which they were obtained by tracing. Off-shell, only the modified equations, of course, have the desired properties of being gauge invariant Lagrangian fields which guarantees their suitability for coupling to conserved sources or alternatively their consistent (path-independent) propagation (Hojman *et al* 1976). The identically vanishing divergence of the left-hand side of these equations constitute source constraints on ψ_μ and $h_{\mu\nu}$, respectively, which can be re-expressed in terms of the field strengths, for comparison with $\partial_\mu(\partial_\nu F^{\mu\nu})\equiv 0$, as

$$\gamma^\nu(\partial^\mu f_{\mu\nu})-\tfrac{1}{2}\not\partial\gamma^\nu(\gamma^\mu f_{\mu\nu})\equiv 0 \qquad (\partial^\mu C^\lambda{}_{\mu\lambda\nu})-\tfrac{1}{2}(\partial_\nu C^{\lambda\rho}{}_{\lambda\rho})\equiv 0. \tag{25}$$

The parentheses enclose factors whose vanishing by (contracted) field strength properties is not of course required for the validity of the identities.

The extension of the above procedures to establish the gauge invariant Lagrangian wave equations for the helicity $\frac{5}{2}$ potential is straightforward. It is also clear that, with appropriate generalisation, the method ought to be applicable to the arbitrary helicity field strengths established in § 6 to give a uniform and direct method for establishing the free field Lagrangian wave equations of Fronsdal (1978) and Fang and Fronsdal (1978), which have also been displayed in a somewhat different form by de Wit and Freedman (1980), Berends *et al* (1985a, b) and Burgers (1985), and in the fermionic case, in particular, by Aragone and Deser (1980a, b) using a vierbein formalism. This extension to arbitrary helicity where new effects arise, such as doubly traceless potentials for spin $\geqslant 4$, will be discussed in subsequent papers (Doughty and Collins 1986a, b, Collins and Doughty 1986). The first of these papers also comments on the applicability of our formalism to the massive case.

8. Conclusion

We have demonstrated how a great many of the properties of chiral and (anti)self-dual Poincaré fields may be formulated using primarily Pauli matrix algebra in an arbitrary unitary representation. In particular, for spin $\geqslant 1$ one may very simply convert bosonic and fermionic Weyl spinors satisfying Weyl's equations into corresponding (anti)self-dual tensors and tensor-spinors to establish complete sets of symmetries for both of the latter. Furthermore, the procedure for establishing these symmetries may be carried out by methods which are essentially identical for all spins j. The differences are simply those which distinguish bosonic from fermionic fields and the high spin ($j > \frac{3}{2}$) from the low spin cases where some of the symmetries are vacuous.

The systematic treatment of all spins has permitted a uniform derivation and direct comparison of the features of the gauge invariant wave equations and souce constraints for the spin-1, $\frac{3}{2}$ and 2 Lagrangian gauge potentials of Maxwell, Rarita-Schwinger and Fierz-Pauli. It also strongly suggests that such a direct relation between, on the one hand, Weyl spinors and their equivalent tensor or tensor-spinor field strengths and, on the other, Lagrangian fields with gauge invariant differential wave equations of appropriate (first or second) order, can be established for arbitrary spin.

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Appendix. Covariant Pauli algebra defining relations and identities

Many parity conjugate equations, obtainable by interchanging $+\leftrightarrow-$ on $\overset{\pm}{\sigma}{}^{\mu}$ and $\overset{\pm}{\sigma}{}^{\mu\nu}$ with reversal in sign of terms in $\varepsilon^{\mu\nu\lambda\rho}$ or the dual ($\sim$), have been omitted.

Defining relations:

$$\overset{\pm}{\sigma}{}^{\mu} = (\mathbb{1}, \pm\sigma^{k}\} \qquad \overset{\pm}{\sigma}{}^{\dagger}_{\mu} = \overset{\pm}{\sigma}_{\mu} \qquad \mu = 0, 1, 2, 3. \tag{A1}$$

Algebra:

$$\overset{+}{\sigma}{}^{\mu}\overset{-}{\sigma}{}^{\nu} + \overset{+}{\sigma}{}^{\nu}\overset{-}{\sigma}{}^{\mu} = 2\eta^{\mu\nu} \qquad \overset{-}{\sigma}{}^{\mu}\overset{+}{\sigma}{}^{\nu} + \overset{-}{\sigma}{}^{\nu}\overset{+}{\sigma}{}^{\mu} = 2\eta^{\mu\nu}. \tag{A2}$$

Products:

$$\overset{-}{\sigma}^{\mu}\overset{+}{\sigma}^{\nu} = \eta^{\mu\nu} + \tfrac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\lambda\rho}\overset{-}{\sigma}_{\lambda}\overset{+}{\sigma}_{\rho} \tag{A3a}$$

$$\overset{+}{\sigma}^{\mu}\overset{-}{\sigma}^{\nu} = \eta^{\mu\nu} - \tfrac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\lambda\rho}\overset{+}{\sigma}_{\lambda}\overset{-}{\sigma}_{\rho} \tag{A3b}$$

$$\overset{-}{\sigma}^{\mu}\overset{+}{\sigma}^{\nu}\overset{-}{\sigma}^{\lambda} - \overset{-}{\sigma}^{\lambda}\overset{+}{\sigma}^{\nu}\overset{-}{\sigma}^{\mu} = \overset{-}{\sigma}^{\nu}\overset{+}{\sigma}^{\lambda}\overset{-}{\sigma}^{\mu} - \overset{-}{\sigma}^{\mu}\overset{+}{\sigma}^{\lambda}\overset{-}{\sigma}^{\nu} = -2\mathrm{i}\varepsilon^{\mu\nu\lambda\rho}\overset{-}{\sigma}_{\rho} \tag{A4}$$

$$\overset{-}{\sigma}^{\lambda}\overset{+}{\sigma}^{\mu}\overset{-}{\sigma}^{\nu} + \overset{-}{\sigma}^{\nu}\overset{+}{\sigma}^{\mu}\overset{-}{\sigma}^{\lambda} = 2(\eta^{\mu\nu}\overset{-}{\sigma}^{\lambda} + \eta^{\mu\lambda}\overset{-}{\sigma}^{\nu} - \eta^{\nu\lambda}\overset{-}{\sigma}^{\mu}). \tag{A5}$$

Contractions:

$$\overset{+}{\sigma}^{\mu}\overset{-}{\sigma}_{\mu} = 4 = \overset{-}{\sigma}^{\mu}\overset{+}{\sigma}_{\mu} \tag{A6}$$

$$\overset{-}{\sigma}^{\lambda}\overset{+}{\sigma}^{\mu}\overset{-}{\sigma}_{\lambda} = -2\overset{-}{\sigma}^{\mu} \tag{A7}$$

$$\overset{-}{\sigma}^{\lambda}\overset{+}{\sigma}_{\mu}\overset{-}{\sigma}_{\nu}\overset{+}{\sigma}_{\lambda} = 4\eta_{\mu\nu} = \overset{+}{\sigma}^{\lambda}\overset{-}{\sigma}_{\mu}\overset{+}{\sigma}_{\nu}\overset{-}{\sigma}_{\lambda}. \tag{A8}$$

Traces:

$$\mathrm{Tr}(\overset{+}{\sigma}^{\mu}\overset{-}{\sigma}^{\nu}) = 2\eta^{\mu\nu} = \mathrm{Tr}(\overset{-}{\sigma}^{\mu}\overset{+}{\sigma}^{\nu}) \tag{A9}$$

$$\mathrm{Tr}(\overset{+}{\sigma}^{\mu}\overset{-}{\sigma}^{\nu}\overset{+}{\sigma}^{\lambda}\overset{-}{\sigma}^{\rho}) = 2(\eta^{\mu\nu}\eta^{\lambda\rho} + \eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\rho} + \mathrm{i}\varepsilon^{\mu\nu\lambda\rho}). \tag{A10}$$

Completeness relation of $\{\mathbb{1}, \sigma^{k}\}$ with respect to 2×2 complex matrices $\boldsymbol{V}$:

$$\boldsymbol{V} = \tfrac{1}{2}\overset{+}{\sigma}^{\mu}\ \mathrm{Tr}(\boldsymbol{V}\overset{-}{\sigma}_{\mu}) = \tfrac{1}{2}\overset{-}{\sigma}^{\mu}\ \mathrm{Tr}(\overset{+}{\sigma}_{\mu}\boldsymbol{V}). \tag{A11}$$

Defining relations of generators:

$$\overset{-}{\sigma}^{\mu\nu} = \tfrac{1}{2}\mathrm{i}(\overset{+}{\sigma}^{\mu}\overset{-}{\sigma}^{\nu} - \overset{+}{\sigma}^{\nu}\overset{-}{\sigma}^{\mu}) = \overset{-}{\sigma}^{[\mu\nu]} \tag{A12a}$$

$$\overset{+}{\sigma}^{\mu\nu} = \tfrac{1}{2}\mathrm{i}(\overset{-}{\sigma}^{\mu}\overset{+}{\sigma}^{\nu} - \overset{-}{\sigma}^{\nu}\overset{+}{\sigma}^{\mu}) = \overset{+}{\sigma}^{[\mu\nu]}. \tag{A12b}$$

Commutator of $S^{\mu\nu} = \frac{1}{2}\overset{+}{\sigma}^{\mu\nu}$ or $\frac{1}{2}\overset{-}{\sigma}^{\mu\nu}$:

$$[S^{\mu\nu}, S^{\lambda\rho}] = \mathrm{i}(\eta^{\nu\lambda}S^{\mu\rho} + \eta^{\mu\rho}S^{\nu\lambda} - \eta^{\mu\lambda}S^{\nu\rho} - \eta^{\nu\rho}S^{\mu\lambda}) \tag{A13}$$

$$\overset{+}{\sigma}^{\mu}\overset{-}{\sigma}^{\nu} = \eta^{\mu\nu} - \mathrm{i}\overset{-}{\sigma}^{\mu\nu} = \eta^{\mu\nu} + \mathrm{i}\overset{-}{\sigma}^{\nu\mu}. \tag{A14}$$

(Anti)self-duality of $\overset{\pm}{\sigma}^{\mu\nu}$:

$$\mathrm{i}\tilde{\overset{\pm}{\sigma}}{}^{\mu\nu} = \pm\overset{\pm}{\sigma}^{\mu\nu} = \tfrac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\lambda\rho}\overset{\pm}{\sigma}_{\lambda\rho} = -\tfrac{1}{2}\varepsilon^{\mu\nu\lambda\rho}\overset{\mp}{\sigma}_{\lambda}\overset{\pm}{\sigma}_{\rho}. \tag{A15}$$

Products:

$$\overset{+}{\sigma}^{\mu}\overset{+}{\sigma}^{\nu\lambda} = -\varepsilon^{\mu\nu\lambda\rho}\overset{+}{\sigma}_{\rho} + \mathrm{i}\eta^{\mu\nu}\overset{+}{\sigma}^{\lambda} - \mathrm{i}\eta^{\mu\lambda}\overset{+}{\sigma}^{\nu} \tag{A16}$$

$$\overset{-}{\sigma}^{\nu\lambda}\overset{+}{\sigma}^{\mu} = -\varepsilon^{\mu\nu\lambda\rho}\overset{+}{\sigma}_{\rho} - \mathrm{i}\eta^{\mu\nu}\overset{+}{\sigma}^{\lambda} + \mathrm{i}\eta^{\mu\lambda}\overset{+}{\sigma}^{\nu}. \tag{A17}$$

Contractions:

$$\overset{-}{\sigma}^{\lambda}\overset{-}{\sigma}_{\mu\nu}\overset{+}{\sigma}_{\lambda} = 0 = \overset{+}{\sigma}^{\lambda}\overset{+}{\sigma}_{\mu\nu}\overset{-}{\sigma}_{\lambda} \tag{A18}$$

$$\overset{+}{\sigma}^{\mu\nu}\otimes\overset{-}{\sigma}_{\mu\nu} = 0 = \overset{-}{\sigma}^{\mu\nu}\otimes\overset{+}{\sigma}_{\mu\nu} \tag{A19}$$

(where $\otimes$ denotes direct product)

$$\overset{+}{\sigma}^{\mu\nu}\overset{+}{\sigma}_{\mu\nu} = 12 = \overset{-}{\sigma}^{\mu\nu}\overset{-}{\sigma}_{\mu\nu} \tag{A20}$$

$$\overset{\pm}{\sigma}^{\mu\lambda}\overset{\pm}{\sigma}_{\lambda\nu} = 2\mathrm{i}\overset{\pm}{\sigma}^{\mu}{}_{\nu} - 3\delta^{\mu}{}_{\nu}. \tag{A21}$$

Traces:

$$\mathrm{Tr}(\overset{+}{\sigma}_{\mu\nu}) = 0 = \mathrm{Tr}(\overset{-}{\sigma}_{\mu\nu}) \tag{A22}$$

$$\mathrm{Tr}(\overset{+}{\sigma}^{\mu\nu}\overset{+}{\sigma}^{\lambda\rho}) = 2(\eta^{\mu\lambda}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\lambda} + \mathrm{i}\varepsilon^{\mu\nu\lambda\rho}) \tag{A23}$$

$$\mathrm{Tr}(\overset{+}{\sigma}^{\mu\nu}\overset{+}{\sigma}^{\lambda\rho} + \overset{-}{\sigma}^{\mu\nu}\overset{-}{\sigma}^{\lambda\rho}) = 4(\eta^{\mu\lambda}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\lambda}) \tag{A24}$$

Completeness of σ^k and hence of $\overset{\pm}{\sigma}^{\mu\nu}$ for traceless 2×2 matrices V:

$$V=\tfrac{1}{8}\overset{\pm}{\sigma}^{\mu\nu}\,\mathrm{Tr}(V\overset{+}{\sigma}_{\mu\nu}). \tag{A25}$$

(i) Dirac identities and indexed Weyl spinor identities. Explicitly indexed Weyl spinor identities may be obtained from each of the above matrix identities by consistently inserting spinorial row and column indices. For example, to raise and lower spinor indices according to $\chi^A=\varepsilon^{AB}\chi_B$ and $\chi_B=\chi^A\varepsilon_{AB}$ (and similarly for dotted indices) requires $\varepsilon^{AB}\varepsilon_{AC}=\delta^B{}_C$ and we may choose a friendly rep of the Pauli algebra ($\varepsilon=\varepsilon^*$, $\varepsilon^2=-\mathbb{1}$) with $\varepsilon=(\varepsilon^{AB})=(\varepsilon_{\dot U\dot V})$ and $\varepsilon=-\varepsilon^{-1}=-\varepsilon^T=(\varepsilon_{AB})=(\varepsilon^{\dot U\dot V})$. We now require $\psi_+^\dagger\overset{+}{\sigma}^\mu\partial_\mu\psi_+$ to be invariant. Consequently we must assign one dotted and one undottted index to $\overset{+}{\sigma}^\mu$ and arbitrarily assign them by taking $\overset{+}{\sigma}^\mu=(\sigma^{\mu A\dot U})$. Lowering A and $\dot U$ gives $\sigma^\mu{}_{B\dot V}=\sigma^{\mu A\dot U}\varepsilon_{AB}\varepsilon_{\dot U\dot V}=(\varepsilon^{-1}\overset{+}{\sigma}^\mu\varepsilon)_{B\dot V}=(\bar\sigma^{\mu T})_{B\dot V}$, where in the last equality we have used the purely matrix equality from § 2.3. Thus, for consistency, we must index $\bar\sigma^\mu$ according to $(\bar\sigma^\mu)_{\dot VB}=\sigma^\mu{}_{B\dot V}$ leading to

$$\overset{+}{\sigma}^\mu\bar\sigma^\nu=(\overset{+}{\sigma}^\mu)^{A\dot U}(\bar\sigma^\nu)_{\dot UB}=\sigma^{\mu A\dot U}\sigma^\nu{}_{B\dot U}=(\overset{+}{\sigma}^\mu\bar\sigma^\nu)^A{}_B$$

and

$$\bar\sigma^\mu\overset{+}{\sigma}^\nu=(\bar\sigma^\mu)_{\dot UA}(\overset{+}{\sigma}^\nu)^{A\dot V}=\sigma^\mu{}_{A\dot U}\sigma^{\nu A\dot V}=(\bar\sigma^\mu\overset{+}{\sigma}^\nu)_{\dot U}{}^{\dot V}.$$

Finally, as examples of indexing the identities above we list

(A9) $\quad \sigma^{\mu A\dot U}\sigma^\nu{}_{A\dot U}=2\delta^\mu{}_\nu \qquad$ (orthonormality of $\sigma^{\mu A\dot U}$)

(A11) $\quad \sigma^{\mu A\dot U}\sigma_{\mu B\dot V}=2\delta^A{}_B\delta^{\dot U}{}_{\dot V} \qquad$ (completeness of $\sigma^{\mu A\dot U}$)

(A12*a*) $\quad \bar\sigma^{\mu\nu A}{}_B=\tfrac{1}{2}\mathrm{i}(\sigma^{\mu A\dot U}\sigma^\nu{}_{B\dot U}-\sigma^{\nu A\dot U}\sigma^\mu{}_{B\dot U})$

(A19) $\quad \overset{+}{\sigma}^{\mu\nu}{}_{\dot V}{}^{\dot U}\bar\sigma_{\mu\nu}{}^A{}_B=0$

(A25) $\quad \bar\sigma^{\mu\nu AB}\bar\sigma_{\mu\nu CD}=-4(\delta^A{}_C\delta^B{}_D+\delta^A{}_D\delta^B{}_C).$

Dirac identities may be routinely reduced to the above covariant Pauli identities, and their extensions, or constructed from them by using the Weyl or chiral representation in which

$$\gamma^\mu=\begin{pmatrix}0 & \bar\sigma^\mu\\ \overset{+}{\sigma}^\mu & 0\end{pmatrix}\qquad \gamma^{\mu\nu}=\begin{pmatrix}\overset{+}{\sigma}^{\mu\nu} & 0\\ 0 & \bar\sigma^{\mu\nu}\end{pmatrix}\qquad C=\begin{pmatrix}-\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix}\qquad \gamma_5=\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}.$$

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